A GENERALIZATION OF THE PONTRYAGIN-HILL THEOREMS TO PROJECTIVE MODULES OVER PRÜFER DOMAINS

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ABSTRACT. Motivated by the Pontryagin-Hill criteria of freeness for abelian groups, we investigate conditions under which unions of ascending chains of projective modules are again projective. Several extensions of these criteria are proved for modules over arbitrary rings and domains, including a genuine generalization of Hill's theorem for projective modules over Prüfer domains with a countable number of maximal ideals. More precisely, we prove that, over such domains, modules which are unions of countable ascending chains of projective, pure submodules are likewise projective.

1. INTRODUCTION

In the last century, Lev Pontryagin and Paul Hill studied conditions under which torsion-free abelian groups are free. In their investigations, the concept of purity of subgroups was crucial. More precisely, a subgroup H of an abelian group G is *pure* if every equation of the form $kx = a \in H$, with $k \in \mathbb{Z}$, is solvable in H whenever it is solvable in G. Equivalently, solvability in G of each system of equations of the form

$$(1.1) \qquad \sum_{j=1}^{m} k_{ij} x_j = a_i \in H \quad (i = 1, \ldots, n),$$

with every $k_{ij} \in \mathbb{Z}$, implies its solvability in H.

In 1934, Pontryagin proved that a countable, torsion-free abelian group is free if and only if every finite rank, pure subgroup is free [19]. Equivalently, every properly ascending chain of pure subgroups of finite rank is finite. From the proof of this result, it follows that a torsion-free abelian group G is free if there exists an ascending chain

$$(1.2) \qquad 0 = G_0 \hookrightarrow G_1 \hookrightarrow \ldots \hookrightarrow G_n \hookrightarrow \ldots \quad (n < \omega)$$

consisting of pure subgroups of G whose union is equal to G, such that every G_n is free and countable.

Later, in 1970, Hill established that, in order for an abelian group G to be free, it is sufficient to prove that it is the union of a countable ascending chain (1.2) of free, pure subgroups [13]. In other words, Hill proved that the condition of countability on the cardinality of the links G_n in Pontryagin's theorem was superfluous. The proof of this theorem relies on some important facts about commutative groups, one of them being that subgroups of torsion-free abelian groups can be embedded

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in pure subgroups of the same rank. Applications of these criteria may be actually found in a variety of algebraic results [7, 8, 9, 18].

In view of the importance of the Pontryagin-Hill theorems in algebra, it is highly desirable to explore the possibility to generalize these criteria to the more general scenario of projective modules over suitable rings. With that purpose in mind, the present work establishes positive answers in the search for extensions of these theorems. Section 2 of this article is devoted to state some useful results concerning relative divisibility, purity, projectivity and localizations of modules. In section 3, we prove some propositions related to classes of families of sets which are relevant to our investigation, while the last section presents the generalizations of the Pontryagin-Hill criteria to projective modules over the domains of interest, namely, semi-hereditary domains.

2. Preliminaries

By a *ring* we mean a ring with an identity element, and by an *integral domain* or, simply, a *domain*, we understand a commutative ring without divisors of zero. In this context, a *Prüfer domain* is a semi-hereditary domain, that is, a domain in which finitely generated ideals are projective.

Over an integral domain, any two maximal independent subsets of a torsion-free module have the same cardinality, the common cardinal number being called the *rank* of the module. Clearly, if a torsion-free module over a domain has rank at most κ, for some cardinal number κ, then every submodule also has rank at most κ. Moreover, if a torsion-free module over an integral domain is at most κ-generated, then it has rank less than or equal to κ.

2.1. **Pure submodules.** Let R be a ring. A submodule N of an R-module M is *relatively divisible* if the inclusion $N \cap rM \hookrightarrow rN$ holds, for every $r \in R$. Equivalently, solvability in M of equations of the form $rx = a \in N$, with $r \in R$, implies their solvability in N. We say that N is *pure* in M if every finite system of equations

$$(2.1) \qquad \sum_{j=1}^{m} r_{ij} x_j = a_i \in N \quad (i = 1, \ldots, n),$$

with $r_{ij} \in R$, is solvable in N whenever it is solvable in M. Under these circumstances, a short-exact sequence $0 \to N \to M \to Q \to 0$ is RD-exact (respectively, pure-exact) if N is a relatively divisible (respectively, pure) submodule of M. Evidently, purity implies relative divisibility, and they both coincide for modules over Prüfer domains [20]. Moreover, Prüfer domains are the only integral domains for which relative divisibility and purity are equivalent [5].

The conditions of relative divisibility and purity posses many interesting properties. For instance, they are closed under unions of ascending chains of arbitrary lengths. Moreover, the intersection of relatively divisible submodules of a torsion-free module M is again relatively divisible. Thus, for every subset X of M, there exists a smallest relatively divisible submodule of M containing X, called the *relatively divisible hull* of X. For torsion-free modules over Prüfer domains, this submodule of M is called the *purification* of X in M, in view that it coincides with the smallest pure submodule $\langle X \rangle_*$ of M containing X. It is worth noticing that, over integral domains, the relatively divisible hull of a submodule has the same

rank as the submodule itself. For proofs of these and more properties on relative divisibility and purity of modules, refer to Sections I.7 and I.8 of [12].

2.2. **Projective modules.** Some elementary results on projective modules will be needed in the present work. For the sake of simplicity, modules are defined over an integral domain R, unless stated otherwise.

The following result for modules over domains is of utmost importance. It was generalized by Endo [10] to more general settings. Indeed, a more general result states that finitely presented, flat modules over arbitrary rings are projective.

Theorem 1 (Cartier [6]). *Finitely generated, flat modules over integral domains are projective.* □

Let B be a flat module. Our next result uses the fact that an exact sequence $0 \to A \to B \to C \to 0$ is pure-exact if and only if C is flat (see [12], Lemma VI.9.1).

Lemma 2. *Every finite rank, pure submodule of a projective module over an integral domain is a finitely generated, projective module.*

Proof. Let A be a finite rank, pure submodule of a projective module M. Without loss of generality, we can assume that M is a free module and, moreover, that it is of finite rank. This means that M is finitely generated. Now, the purity of A implies that M/A is a finitely generated, flat module. By Theorem 1, A is a direct summand of M and, hence, it is a finitely generated, projective module. □

Lemma 3. *Every countable rank, pure submodule of a projective module over an integral domain is contained in a countably generated submodule.*

Proof. Let A be a countable rank, pure submodule of the projective module M. Then, there exists a free module F which contains M as a direct summand. So, A is pure in F and is contained in a countably generated, free summand N of F. Then, the image of N under the projection homomorphism of F onto M is again countably generated and contains A. □

Corollary 4. *Over integral domains, projective modules of countable rank are countably generated.* □

Corollary 5. *Every countably generated module of projective dimension at most 1 over an integral domain is countably presented.* □

2.3. **Localizations.** Throughout this section, R will represent an integral domain. Here, we obey the tradition of representing the localization of a torsion-free module M at a multiplicatively closed set $S \subseteq R$ by M_S, and the localization of M at the complement of a prime ideal P of R by M_P. Moreover, we denote by $D(M)$ the localization of M with respect to $R \setminus 0$, and the set of maximal ideals of R by $\max R$.

Lemma 6. *Let R have countably many maximal ideals. A torsion-free R-module M is countably generated if and only if the localization of M at every maximal ideal P of R is a countably generated R_P-module.*

Proof. We only need to prove that the given condition is sufficient. For each maximal ideal P of R, choose a countable generating set of the R_P-module M_P, consisting of elements of the form $a_P^n/1$, with $a_P^n \in M$, for every $n < \omega$. Let M' be the

R-submodule of $D(M)$ generated by

$$Y = \bigcup_{P \in \max R} \{a_P^n / 1 : n < \omega\}. \tag{2.2}$$

Obviously, Y is contained in M and, therefore, M' is contained in M as a sub-module. Moreover, for every maximal ideal P of R, we have that $M_P = M'_P$ as R_P-modules; consequently, $M = M'$. □

Proposition 7. *Let R be any integral domain which is not a field, and let S be a countably infinite, multiplicatively closed subset of R which does not contain any unit of R. Then, there exists a countable ascending chain*

$$0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_n \hookrightarrow \ldots \quad (n < \omega) \tag{2.3}$$

of submodules of the R-module $J = R_S$, such that:

    (i) *every M_n is projective,*
    (ii) *none of the M_n is pure in J, and*
    (iii) $J = \bigcup_{n < \omega} M_n$.

Proof. Let $S = \{s_n : n \in \mathbb{Z}^+\}$. For every positive integer n, let M_n be the R-submodule of J given by

$$M_n = \left\langle \frac{1}{s_1 \cdot \ldots \cdot s_n} \right\rangle \cong R. \tag{2.4}$$

Obviously, the sequence $\{M_n\}_{n < \omega}$ forms an ascending chain (2.3) of R-modules satisfying (i) and (iii). In order to establish (ii), it suffices to prove that every M_n is not relatively divisible in M. Indeed, observe that the equation

$$s_{n+1} x = \frac{1}{s_1 \cdot \ldots \cdot s_n} \tag{2.5}$$

is solvable in the torsion-free R-module J, the only solution being

$$x = \frac{1}{s_1 \cdot \ldots \cdot s_n s_{n+1}}. \tag{2.6}$$

However, (2.5) is not solvable in M_n, so that M_n is not pure in M. □

The existence of the set S in Proposition 7 is guaranteed for every integral domain R which is not a field. For instance, S can be the set of all positive integer powers of a nonzero, non-invertible element of R.

Proposition 8. *Let R be an integral domain which is not a field, and let S be a countably infinite, multiplicatively closed subset of R which does not contain any unit of R. Then, there exists a pure-exact sequence*

$$0 \to H \to F \to J \to 0, \tag{2.7}$$

with H and F free R-modules of countably infinite rank, and $J = R_S$.

Proof. Proposition 7 guarantees the existence of a countable ascending chain (2.3) of projective R-modules whose union equals J, and it follows by Auslander's lemma [1] that $\mathrm{pd}_R J \leq 1$. However, J is an infinitely generated localization of R, so it cannot be projective. As a consequence, J is a flat module of projective dimension 1; moreover, since it is countably generated, then J is countably presented by Corollary 5. Thus, there exists a pure-exact sequence

$$0 \to P \to F \to J \to 0 \tag{2.8}$$

of R-modules, with P a countably generated, projective module, and F a free module of countably infinite rank. By a well-known result of Eilenberg (exercise 1.1 in Chapter VI of [12]), there exists a free module H of countably infinite rank, such that $P \oplus H \cong H$. Therefore, the induced sequence

$$(2.9) \qquad\qquad 0 \to P \oplus H \to F \oplus H \to J \to 0$$

is pure-exact. Notice finally that $F \oplus H$ is isomorphic to F, whence the conclusion follows. □

It is important to keep in mind that the R-module J in Proposition 8 is not projective. Also, it is useful to point out that localizations of Prüfer domains are again Prüfer domains; particularly, localizations of Prüfer domains at prime ideals are valuation domains [12].

3. Families of modules

A $G(\aleph_0)$-*family* of a module M over a ring R is a family $\mathcal{B}$ consisting of submodules of M, with the following properties:

   (i) $0, M \in \mathcal{B}$,
   (ii) $\mathcal{B}$ is closed under unions of ascending chains of arbitrary lengths, and
   (iii) for every $A_0 \in \mathcal{B}$ and every countable set $H \subseteq M$, there exists $A \in \mathcal{B}$ containing A_0 and H, such that A/A_0 is countably generated.

Clearly, an intersection of a countable number of $G(\aleph_0)$-families of submodules of M is again a $G(\aleph_0)$-family of submodules of M. Examples of $G(\aleph_0)$-families are axiom-3 families. By an *axiom*-3 family of M we mean a family $\mathcal{B}$ of submodules of M satisfying (i) and (iii) above, plus the property:

   (ii)′ $\mathcal{B}$ is closed under arbitrary sums.

A $G(\aleph_0)$-family of submodules of M is a *tight system* if, in addition, it satisfies:

   (iv) for every $A \in \mathcal{B}$, $\mathrm{pd}_R A \leq 1$ and $\mathrm{pd}_R(M/A) \leq 1$.

It is worth mentioning that every module has a $G(\aleph_0)$-family of submodules, namely, the collection of all its submodules. However, the existence of a $G(\aleph_0)$-family of pure submodules is not guaranteed in general for every R-module, not even when R is an integral domain. However, over a valuation domain, a torsion-free module of projective dimension less than or equal to 1 has a tight system of pure submodules [2].

In the proof of the following result, we use ideas previously applied in the investigation of the freeness of abelian groups [14, 15], and Butler groups of infinite rank [3, 4].

Theorem 9. *Let R be a Prüfer domain with a countable number of maximal ideals. Every torsion-free R-module of projective dimension at most equal to 1 has a $G(\aleph_0)$-family of pure submodules.*

Proof. Let $\{P_n\}_{n \in \mathbb{Z}^+}$ be all the maximal ideals of R, and let M be a torsion-free R-module of projective dimension at most equal to 1. For every positive integer n, the module M_{P_n} is torsion-free of projective dimension at most equal to 1 over the valuation domain R_{P_n} and, so, it has a $G(\aleph_0)$-family $\mathcal{B}_n$ consisting of pure R_{P_n}-submodules. Clearly, every $\mathcal{B}_n$ is also a family of pure submodules of M_{P_n} when considered as an R-module.

Let

(3.1) $$\mathcal{B}'_n = \mathcal{B}_n \cap M = \{A \cap M : A \in \mathcal{B}_n\} \quad (n \in \mathbb{Z}^+).$$

Obviously, every $\mathcal{B}'_n$ is a family of pure R-submodules of M which is closed under unions of ascending chains, such that $0, M \in \mathcal{B}'_n$. Therefore, the intersection $\mathcal{B}$ of all families $\mathcal{B}'_n$ is a family of pure R-submodules of M satisfying properties (i) and (ii) of a $G(\aleph_0)$-family, and we only need to prove (iii).

Let $B_0 \in \mathcal{B}$, and let $H_0 \subseteq M$ be a countable set. For every positive integer n, there exists $A_n^0 \in \mathcal{B}_n$ such that $B_0 = A_n^0 \cap M$. Our argument hinges on the construction of a countable ascending chain

(3.2) $$B_0 \hookrightarrow B_1 \hookrightarrow \ldots \hookrightarrow B_m \hookrightarrow \ldots \quad (m < \omega)$$

of submodules of the R-module M, such that:

   (a) H_0 is contained in B_1, and
   (b) B_{m+1} has countable rank over B_m, for every $m < \omega$.

More precisely, for every $m < \omega$, B_{m+1} is the union of a countable ascending chain

(3.3) $$B_m = B_0^{m+1} \hookrightarrow B_1^{m+1} \hookrightarrow \ldots \hookrightarrow B_k^{m+1} \hookrightarrow \ldots \quad (k < \omega)$$

of pure R-submodules of M, satisfying the following properties, for every $k < \omega$:

   (c_m) $B_k^{m+1} = A_k^{m+1} \cap M$ with $A_k^{m+1} \in \mathcal{B}_k$,
   (d_m) A_k^{m+1} is a countably generated R_{P_k}-module over A_k^0, and
   (e_m) B_k^{m+1} has countable rank over B_0.

Assume that the links of the finite ascending chain

(3.4) $$B_0 \hookrightarrow B_1 \hookrightarrow \ldots \hookrightarrow B_m$$

have been constructed as desired, for some $m < \omega$. Moreover, let $n < \omega$ and assume that the links of countable ascending chain (3.3) have also been constructed as required, for every $k \le n$. Choose a complete set of representatives G_{n+1}^m of a countable generating system of the $R_{P_{n+1}}$-module A_{n+1}^m modulo A_{n+1}^0, and a complete set of representatives H_n^{m+1} of a maximal independent system of the R-module B_n^{m+1} modulo B_0. Then, there exists $A_{n+1}^{m+1} \in \mathcal{B}_{n+1}$ containing both A_{n+1}^0 and $G_{n+1}^m \cup H_n^{m+1} \cup H_0$, such that A_{n+1}^{m+1} is a countably generated $R_{P_{n+1}}$-module over A_{n+1}^0. (It is worth noticing here that this procedure ensures that B_1^1 will be a module of countable rank over B_0, which contains H_0.)

Let $B_{n+1}^{m+1} = A_{n+1}^{m+1} \cap M$. Properties (c_m) and (d_m) are then satisfied for $k = n+1$. Moreover, since the R-module A_{n+1}^{m+1} has countable rank over A_{n+1}^0, then

(3.5) $$\frac{B_{n+1}^{m+1}}{B_0} \cong \frac{A_{n+1}^0 + (A_{n+1}^{m+1} \cap M)}{A_{n+1}^0} \hookrightarrow \frac{A_{n+1}^{m+1}}{A_{n+1}^0}$$

is also a countable rank R-module, so that property (e_m) is also satisfied for $k = n + 1$. By induction, we construct the countable ascending chain (3.3) with the desired properties, and let B_{m+1} be the union of the links of such chain.

Inductively, we construct a countable ascending chain (3.2) with the desired properties. Then, $B = \bigcup_{m<\omega} B_m$ is an R-submodule of M which contains B_0 and H_0. Moreover,

(3.6) $$B = \bigcup_{m<\omega} \bigcup_{k<\omega} (A_k^m \cap M) = \left(\bigcup_{m<\omega} A_n^m \right) \cap M \quad (n \in \mathbb{Z}^+).$$

The fact that $\bigcup_{m<\omega} A_n^m \in \mathcal{B}_n$, for every index n, implies that $B \in \mathcal{B}$.

In order to prove that B/B_0 is a countably generated R-module, observe that exactness of the localization functors yields that

$$(3.7) \qquad \left(\frac{B}{B_0}\right)_{P_n} = \frac{\left[\bigcup_{m<\omega}(A_n^m \cap M)\right]_{P_n}}{(A_n^0 \cap M)_{P_n}} = \frac{\bigcup_{m<\omega} A_n^m}{A_n^0} \quad (n < \omega)$$

is a countably generated R_{P_n}-module. Lemma 6 implies now that B/B_0 is countably generated. $\qquad \square$

For the rest of this section, we will assume that M is a torsion-free module over a Prüfer domain, for which there exists a countable ascending chain (2.3) of submodules, such that the following properties are satisfied:

$\mathbf{P}_1$ every M_n is relatively divisible in M,

$\mathbf{P}_2$ every M_n has a $G(\aleph_0)$-family $\mathcal{B}_n$ of relatively divisible submodules,

$\mathbf{P}_3$ every factor module M_{n+1}/M_n has a $G(\aleph_0)$-family $\mathcal{C}_n$ of relatively divisible submodules, and

$\mathbf{P}_4$ $M = \bigcup_{n<\omega} M_n$.

Lemma 10. *For every $n < \omega$, the collection*

$$(3.8) \qquad \mathcal{B}'_n = \left\{ A \in \mathcal{B}_n : \frac{(A + M_j) \cap M_{j+1}}{M_j} \in \mathcal{C}_j, \text{ for every } j < n \right\}$$

is a $G(\aleph_0)$-family of relatively divisible submodules of M_n.

Proof. The collection $\mathcal{B}'_n$ clearly satisfies conditions (i) and (ii) of a $G(\aleph_0)$-family of submodules of M_n. So, let $A_0 \in \mathcal{B}'_n$, and let $H_0 \subseteq M_n$ be a countable set. We will construct an ascending chain

$$(3.9) \qquad A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_m \hookrightarrow \ldots \quad (m < \omega)$$

of submodules of M_n, such that:

(a) A_1 contains H_0,

(b) $A_i \in \mathcal{B}_n$, for every $i < \omega$,

(c) A_{i+1}/A_i is countably generated, for every $i < \omega$, and

(d) for every $i < \omega$ and $j < n$, there exists a module $K_{i,j}/M_j \in \mathcal{C}_j$, such that

$$(3.10) \qquad \frac{(A_i + M_j) \cap M_{j+1}}{M_j} \hookrightarrow \frac{K_{i,j}}{M_j} \hookrightarrow \frac{(A_{i+1} + M_j) \cap M_{j+1}}{M_j}.$$

To start with, we let $K_{0,j} = (A_0 + M_j) \cap M_{j+1}$, for every $j < n$. In general, suppose that, for some $m < \omega$, the links of the finite chain

$$(3.11) \qquad A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_m$$

have been constructed as desired. Since A_m/A_0 is countably generated, then we can choose a countable set of representatives $X_j \subseteq M_{j+1}$ of a maximal independent system of the purification of $(A_m + M_j) \cap M_{j+1}$ modulo $(A_0 + M_j) \cap M_{j+1}$, for every $j < n$. For each such j, there exists $K_{m,j}/M_j \in \mathcal{C}_j$ containing both the module $[(A_0 + M_j) \cap M_{j+1}]/M_j$ and the quotient set X_j modulo M_j, such that

$$(3.12) \qquad \frac{K_{m,j}/M_j}{[(A_0 + M_j) \cap M_{j+1}]/M_j} \cong \frac{K_{m,j}}{(A_0 + M_j) \cap M_{j+1}}$$

is countably generated. Thus, it is readily checked that the first inclusion of (3.10) is satisfied for $i = m$.

For every $j < n$, let $H_j^m \subseteq K_{m,j}$ be a complete set of representatives of a countable generating system of $K_{m,j}$ modulo $(A_0 + M_j) \cap M_{j+1}$. Then, there exists $A_{m+1} \in \mathcal{B}_n$ containing both A_m and $H_0 \cup (\bigcup_{j<n} H_j^m)$, such that A_{m+1}/A_m is countably generated. Our construction guarantees that

$$(3.13) \qquad \frac{K_{m,j}}{(A_0 + M_j) \cap M_{j+1}} \hookrightarrow \frac{(A_{m+1} + M_j) \cap M_{j+1}}{(A_0 + M_j) \cap M_{j+1}} \quad (j < n),$$

so that the second inclusion of (3.10) is satisfied also for $i = m$.

Inductively, we construct an ascending chain (3.9) of submodules of M_n, satisfying (a), (b), (c) and (d) above. Then, the module $A = \bigcup_{m<\omega} A_m$ is a member of $\mathcal{B}_n$ which contains A_0 and H_0, and is countably generated over A_0. Moreover, notice that property (d) implies that

$$(3.14) \qquad \frac{(A + M_j) \cap M_{j+1}}{M_j} = \bigcup_{i<\omega} \frac{(A_i + M_j) \cap M_{j+1}}{M_j} = \bigcup_{i<\omega} \frac{K_{i,j}}{M_j} \in \mathcal{C}_j \quad (j < n).$$

Consequently, $A \in \mathcal{B}_n'$, as desired. $\qquad \square$

Next, we construct a $G(\aleph_0)$-family of relatively divisible submodules of M.

Lemma 11. *The collection*

$$(3.15) \qquad \mathcal{B} = \{A \hookrightarrow M : A \cap M_n \in \mathcal{B}_n', \text{ for every } n < \omega\}$$

is a $G(\aleph_0)$-family of relatively divisible submodules of M.

Proof. Again, conditions (i) and (ii) in the definition of a $G(\aleph_0)$-family are obvious. So, let $A_0 \in \mathcal{B}$, and let H_0 be a countable subset of M. We will construct a countable ascending chain

$$(3.16) \qquad A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_n \hookrightarrow \ldots \quad (n < \omega)$$

of submodules of M, such that:

    (a) A_1 contains H_0, and
    (b) the factor module A_n/A_0 is countably generated, for every $n < \omega$.

More precisely, for every $n < \omega$, the module A_n is the union of a countable ascending chain

$$(3.17) \qquad 0 = A_0^n \hookrightarrow A_1^n \hookrightarrow \ldots \hookrightarrow A_k^n \hookrightarrow \ldots \quad (k < \omega)$$

of submodules of M, for which the following properties are satisfied:

    (c_n) $A_k^n \in \mathcal{B}_k'$, for every $k < \omega$,
    (d_n) A_k^n is countably generated over $A_0 \cap M_k$, for every $k < \omega$, and
    (e_n) $A_k^n \hookrightarrow A_n \cap M_k \hookrightarrow A_k^{n+1}$, for every $k < \omega$.

Let $A_k^0 = A_0 \cap M_k$, for every $k < \omega$. Obviously, A_0 is the union of chain (3.17) with $n = 0$, and properties (c_0) and (d_0) are satisfied, as well as the first inclusion of (e_0). So, let m be now any nonnegative integer, and suppose that the links of chain (3.16) have been constructed as needed, for every $n \leq m$. Condition (b) implies that the module

$$(3.18) \qquad \frac{A_m \cap M_k}{A_0 \cap M_k} \cong \frac{A_0 + (A_m \cap M_k)}{A_0} \hookrightarrow \frac{A_m}{A_0} \quad (k < \omega)$$

has countable rank. Choose then a countable set $Y_k^m \subseteq M_k$ of representatives of a maximal independent system of the purification of $(A_m \cap M_k)/(A_0 \cap M_k)$ in $M/(A_0 \cap M_k)$. Clearly, there exists $B_k^m \in \mathcal{B}_k'$ containing both $A_0 \cap M_k$ and Y_k^m, such that B_k^m is countably generated over $A_0 \cap M_k$; so, we may fix a complete set of representatives $H_k^m \subseteq B_k^m$ of a countable generating set of B_k^m modulo $A_0 \cap M_k$. Moreover, since B_k^m is relatively divisible in M, then

$$(3.19) \qquad \left\langle \frac{A_m \cap M_k}{A_0 \cap M_k} \right\rangle_* \hookrightarrow \frac{B_k^m}{A_0 \cap M_k} \hookrightarrow \frac{M_k}{A_0 \cap M_k} \quad (k < \omega).$$

Let k be a nonnegative integer. Assume that the links of the finite chain

$$(3.20) \qquad 0 = A_0^{m+1} \hookrightarrow A_1^{m+1} \hookrightarrow \ldots \hookrightarrow A_k^{m+1}$$

have been constructed as required, and let $X_k \subseteq M_k$ be a complete set of representatives of a countable generating system of A_k^{m+1} modulo $A_0 \cap M_k$. Lemma 10 implies that there exists $A_{k+1}^{m+1} \in \mathcal{B}_{k+1}'$ which contains $A_0 \cap M_{k+1}$ and the countable set $X_k \cup H_{k+1}^m \cup (H_0 \cap M_{k+1})$, such that A_{k+1}^{m+1} is countably generated over $A_0 \cap M_{k+1}$.

Inductively, we can construct a countable ascending chain (3.17), with $n = m+1$, and define the module A_{m+1} as the union of the links of (3.17), in such way that conditions (c_{m+1}), (d_{m+1}) and (e_{m+1}) be satisfied. Moreover, our construction guarantees that A_m is contained in A_{m+1}. Furthermore, the fact that A_{m+1}/A_0 is countably generated follows from the isomorphism

$$(3.21) \qquad \frac{A_k^{m+1}}{A_0 \cap M_k} \simeq \frac{A_0 + (A_k^{m+1} \cap M_k)}{A_0} \quad (k < \omega).$$

Indeed, the left-hand side of (3.21) is countably generated by properties (d_{m+1}), so that the union over all indexes $k < \omega$ of the modules in the right-hand side is countably generated, too.

By induction, a countable ascending chain (3.16) satisfying properties (a) and (b) is constructed. The module $A = \bigcup_{n<\omega} A_n$ contains A_0 and H_0, and is countably generated over A_0. Moreover, properties (c_n) and (e_n) yield that

$$(3.22) \qquad A \cap M_k = \bigcup_{n<\omega} (A_n \cap M_k) = \bigcup_{n<\omega} A_k^n \in \mathcal{B}_k' \quad (k < \omega).$$

Consequently, $A \in \mathcal{B}$. $\qquad\qquad\square$

Lemma 12. *For every $A \in \mathcal{B}$ and $n < \omega$, $A + M_n$ is relatively divisible in M.*

Proof. It is enough to prove that $(A + M_n) \cap M_k$ is relatively divisible in M, for every $k > n$. First of all, observe that $A \cap M_{n+1} \in \mathcal{B}_{n+1}'$, whence it follows that $(A + M_n) \cap M_{n+1}$ is likewise relatively divisible in M, for every $n < \omega$. So, our claim is true for $k = n + 1$.

Assume now that $(A + M_n) \cap M_k$ is relatively divisible in M, for some $k > n$. The modular law and the isomorphism theorem yield that

$$(3.23) \qquad \frac{(A + M_k) \cap M_{k+1}}{(A + M_n) \cap M_{k+1}} \simeq \frac{M_k}{(A + M_n) \cap M_k}.$$

The last module in (3.23) is torsion-free, so that the first module is also torsion-free. This implies that $(A+M_n) \cap M_{k+1}$ is relatively divisible in $(A+M_k) \cap M_{k+1}$. Finally, since $(A+M_k) \cap M_{k+1}$ is relatively divisible in M, we conclude that $(A+M_n) \cap M_{k+1}$ is also relatively divisible in M. Our claim follows now by induction. $\qquad\square$

4. A GENERALIZATION OF HILL'S THEOREM

In this section, we use the generalization of Pontryagin's criterion of freeness to projective modules, presented as Theorem 1.3, Chapter XVI in [12], in order to provide a generalization of Hill's criterion of freeness. Particularly, we use the fact that a countable rank, torsion-free module over a Prüfer domain is projective if and only if every finite rank, pure submodule is projective.

Beforehand, it is important to mention that the problem of generalizing Hill's criterion was attacked previously in [12] (see Theorem 1.4, Chapter XVI). However, the proof of that version of Hill's theorem for projective modules is wrong, one serious problem being that it is not generally true that the modules $U_1 \cap M_\nu$ in the proof of Lemma XVI.1.6 have rank less than or equal to κ.

The following is our generalization of Hill's theorem to projectivity of modules over Prüfer domains.

Theorem 13. *A module M over a Prüfer domain is projective if there exists a countable ascending chain*

$$(4.1) \qquad 0 = M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_n \hookrightarrow \ldots \quad (n < \omega)$$

of submodules of M, such that:

(i) *every M_n is projective,*
(ii) *every M_n is pure in M,*
(iii) *every factor M_{n+1}/M_n admits a $G(\aleph_0)$-family $\mathcal{C}_n$ of pure submodules, and*
(iv) *$M = \bigcup_{n<\omega} M_n$.*

In the following discussion, assume the hypotheses of Theorem 13. By Kaplansly's theorem on decomposition of projective modules over Prüfer domains, every module M_n can be written as the direct sum of finitely generated modules M_α^n, for α in a set of indices Ω_n. Clearly, the set

$$(4.2) \qquad \mathcal{B}_n = \{A \hookrightarrow M_n : A = \oplus_{\alpha \in \Lambda} M_\alpha^n, \text{ for some } \Lambda \subseteq \Omega\}$$

is an axiom-3 family of direct summands of M_n, for every $n < \omega$. By Lemma 10, the collection $\mathcal{B}_n'$ given by (3.8) is a $G(\aleph_0)$-family of relatively divisible submodules of M_n. Moreover, Lemma 11 and Lemma 12 state that the family $\mathcal{B}$ provided by (3.15) is a $G(\aleph_0)$-family of relatively divisible submodules of M, such that for every $A \in \mathcal{B}$ and every $n < \omega$, $A + M_n$ is relatively divisible in M.

Lemma 14. *For every $A \in \mathcal{B}$, finite rank, pure submodules of M/A are projective.*

Proof. Let D be a submodule of M containing A, such that D/A is a finite rank, pure submodule of M/A. Choose a maximal independent system $\{d_i + A : i = 1, \ldots, n\}$ of D/A, and let $S = \{d_1, \ldots, d_n\} \subseteq D$. Let $k \in \mathbb{Z}^+$ be such that $S \subseteq M_k$. Since $A + M_k$ is pure, $D \hookrightarrow A + M_k$. So, $D/A \hookrightarrow (A + M_k)/A \cong M_k/(A \cap M_k)$, which is projective. By Lemma 2, D/A is likewise projective. $\square$

We are now in a position to provide a proof of our main result.

Proof of Theorem 13. Let α be any nonzero ordinal, and let

$$(4.3) \qquad 0 = A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_\gamma \hookrightarrow A_{\gamma+1} \hookrightarrow \ldots \quad (\gamma < \alpha)$$

be an ascending chain of modules in $\mathcal{B}$, such that every factor module $A_{\gamma+1}/A_\gamma$ is projective. If α is a limit ordinal, we let $A_\alpha = \bigcup_{\gamma<\alpha} A_\gamma$. Otherwise, if there exists

$x \in M \setminus A_{\alpha-1}$, we can pick a module $A_\alpha \in \mathcal{B}$ which contains x and $A_{\alpha-1}$, such that $A_\alpha/A_{\alpha-1}$ has countable rank. By Lemma 14, finite rank, pure submodules of the torsion-free module $A_\alpha/A_{\alpha-1}$ are projective, so that $A_\alpha/A_{\alpha-1}$ itself is projective. In such way, transfinite induction provides a continuous, well-ordered, ascending chain

$$(4.4) \qquad 0 = A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_\alpha \hookrightarrow A_{\alpha+1} \hookrightarrow \ldots \quad (\alpha < \tau)$$

of submodules of M, satisfying the hypotheses of Lemma XVI.1.1 in [12]. Consequently, M is projective. $\qquad \square$

Corollary 15. *Let R be a Prüfer domain with a countable number of maximal ideals. An R-module M is projective if there exists a countable ascending chain (4.1) of projective, pure submodules of M, such that $M = \bigcup_{n<\omega} M_n$.*

Proof. For every $n < \omega$, M_{n+1} is torsion-free and contains M_n as a pure submodule, so M_{n+1}/M_n is a torsion-free module of projective dimension at most 1. Then, every factor module of (4.1) admits a $G(\aleph_0)$-family of pure submodules by Theorem 9. By Theorem 13, M is projective. $\qquad \square$

Since valuation domains have a unique maximal ideal, Corollary 15 can be obviously improved as follows.

Corollary 16. *Let M be a module over a valuation domain. If there exists a countable ascending chain (4.1) of free, pure submodules of M whose union is equal to M, then M itself is free.* $\qquad \square$

Let λ be an infinite cardinal number. We say that a module is *λ-free* if each of its submodules of rank less than λ can be embedded in a free, pure submodule.

Corollary 17. *Let λ be an infinite cardinal with co-finality equal to ω. Every torsion-free, λ-free module of rank λ over a valuation domain is free.*

Proof. Let M be a torsion-free, λ-free module over a valuation domain, let $X = \{a_\alpha \in M : \alpha < \lambda\}$ be a maximal independent set in M, and let

$$(4.5) \qquad \lambda_0 < \lambda_1 < \cdots < \lambda_n < \ldots \quad (n < \omega)$$

be a sequence of ordinals whose union is λ. For every $n < \omega$, the purification in M of the set $\{a_\alpha \in M : \alpha < \lambda_n\}$ has rank less than λ and, so, it is contained in a free, pure submodule M_n of M. Obviously, the modules $\{M_n\}_{n<\omega}$ may be chosen to form a countable ascending chain of submodules of M whose union contains X. The conclusion of this result follows now from Corollary 16. $\qquad \square$

As a closing remark, it is worth mentioning that, as of now, we still do not know whether it is possible to extend Theorem 13 to projectivity of modules over integral domains in general.

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